Twisted Yang - Baxter equations for linear quantum (super)groups

1996 J. Phys. A: Math. Gen. 29 6903

(http://iopscience.iop.org/0305-4470/29/21/021)

J. Phys. A: Math. Gen. **29** (1996) 6903–6910. Printed in the UK

Twisted Yang–Baxter equations for linear quantum (super)groups

A P Isaev†

Dipartimento di Fisica, Universitá di Pisa, Piazza Torricelli 2, 56100 Pisa, Italy

Received 9 November 1995

Abstract. We consider the modified (or twisted) Yang–Baxter equations for the $SL_q(N)$ groups and $SL_q(N|M)$ supergroups. The general solutions for these equations are presented in the case of the linear quantum (super)groups. The introduction of spectral parameters in the twisted Yang–Baxter equation and its solutions are also discussed.

1. Introduction

Recently various types of modified Yang–Baxter equation (mYBE) have been considered. First of all such mYBEs appeared in investigations on special exchange algebras [1]. Another one was explored [2] in the context of the construction of new integrable lattice models which generalize the $SL(3)$ (and in general the $SL(N)$) chiral Potts model. After a series of papers (see [3]) devoted to the solutions of the tetrahedron equation, a similar modification, but now for the 3D analogues of the YBE, has been used for a construction of new integrable 3D lattice theories [4]. Moreover, a variant of the YBE has been found as certain cubic relations (for R-matrices being a special set of the quantum 6j symbols) which express the consistency of a quadratic algebra for elements of the matrix generating a set of Clebsch–Gordon coefficients [5–7]. It is interesting that this modification coincides with that considered in [1] and the corresponding mYBE and R-matrices essentially depend on the phase space coordinates. Note that the same dependence occures for the classical r-matrices (which are called dynamical r-matrices) in Calogero–Moser type models (see [8, 9] and references therein). On the other hand, we recall that the q-analogues of the 6j symbols (or Racah coefficients) give the braiding/fusing matrices expressing the property of crossing symmetry for four-point conformal blocks in 2D conformal field theories (see, e.g., [10]). Finally we stress that the analogous 'twisted' YBE has also been proposed in the context of quasi-Hopf algebras [11].

In this paper we investigate the mYBEs that appeared in [1] as consistency relations for exchange matrices and that were considered in [5–7] as some relations for the $SL_q(2)$ 6j symbols. Here, in the cases of the $SL_q(N)$ and $SL_q(N|M)$ (super)groups, we present the explicit general solutions $R(p)$ for such mYBEs. Then we show how one can generalize these mYBEs and their solutions by introducing spectral parameters and also present the Yangian-type limits for these solutions. Our conjecture is that after introducing the spectral parameter we obtain some objects related to the quantum affine Kac–Moody algebras.

† Permanent address: Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow Region, 141 980, Russia.

0305-4470/96/216903+08$19.50

2. Quantum deformations of dynamical systems on cotangent bundles of Lie groups or Alekseev–Faddeev toy models

At the beginning, to introduce the objects which will be under consideration, we recall some facts from [6]. It is known [12] that apparently all finite-dimensional integrable models, like the Toda chain or Calogero particles, can be considered as systems which can be obtained by the Hamiltonian reduction of geodesic motion on the cotangent bundles of some Lie groups G (with Lie algebras $\mathcal{G}$) and described by the Lagrangians

$$\mathcal{L}(t) = \left\langle L \left| \frac{\mathrm{d}}{\mathrm{d}t} g\, g^{-1} \right.\right\rangle - \frac{1}{2}\langle L|L\rangle + \langle L - \mu^L|\phi\rangle + \langle g^{-1}Lg - \mu^R|\psi\rangle \tag{1}$$

where t is time, $g(t) \in G$, $L(t)$ and constant elements $\mu^{L,R}$ belong to the space $\mathcal{G}^*$ dual to the Lie algebra $\mathcal{G}$, terms with $\phi,\ \psi \in \mathcal{G}$ define the momentum mappings, $\phi,\ \psi$ are simply Lagrange multipliers, and $\langle\cdot|\cdot\rangle$ is a paring of $\mathcal{G}$ and $\mathcal{G}^*$. We also identify $\mathcal{G}$ and $\mathcal{G}^*$ through the invariant Killing metric. The explicit choice of the group G, multipliers ϕ, ψ and elements $\mu^{L,R}$ specifies the dynamical system. If we consider the case for which we can take the matrix representation for $\mathcal{G}$ and $\mathcal{G}^*$ such that the pairing will be defined via the operator Tr ($\langle A|B\rangle \to Tr(AB)$), then, one can find the equations of motion from the Lagrangian (1) and prove that the quantities $I_n = Tr(L^n)$ are integrals of motion. Thus, for appropriate momentum mappings we can expect that the system with Lagrangian (1) yields an example of the integrable model. From the Lagrangian (1) we find the following Poisson brackets (see, e.g., [6]):

$$\begin{aligned} &\{g^1, g^2\} = 0 \\ &\{L^1,\ g^2\} = C\, g^2 \\ &\{L^1,\ L^2\} = -\tfrac{1}{2}[C,\ L^1 - L^2] \end{aligned} \tag{2}$$

where as usual $g^1 = g \otimes 1,\ L^2 = 1 \otimes L, \ldots$ and $C = t_a \otimes t_b\, \eta^{ab}$ is an ad-invariant tensor (the η^{ab} define the Killing metric and the t_a form the basis for Lie algebra $\mathcal{G}$). Then, for the case $G = SL(N, C)$ (actually for $SU(N)$), the diagonalization of the left L and right $g^{-1}Lg$ momenta can be considered:

$$L = uPu^{-1} \qquad g^{-1}Lg = v^{-1}Pv \tag{3}$$

and this leads to the diagonalization of the group element g:

$$g = u\, Q^{-1}\, v\,. \tag{4}$$

Here we have used

$$P = -\frac{\mathrm{i}}{2}\mathrm{diag}\{p_1,\ p_2, \ldots, p_N\} \qquad \sum_{i=1}^{N} p_i = 0 \tag{5}$$

$$Q = \mathrm{diag}\{\exp(\mathrm{i}x_1),\ \exp(\mathrm{i}x_2), \ldots, \exp(\mathrm{i}x_N)\} \qquad \sum_{i=1}^{N} x_i = 0 \tag{6}$$

and the matrices $u,\ v$ belong to the homogeneous space G/H where H is a Cartan subgroup associated with P.

In the papers [6, 7, 13] it was shown that Poisson structure (2), in terms of the new variables $\{u,\ v,\ P,\ Q\}$, acquires the form

$$\{u^1,\ u^2\} = -u^1\, u^2\, r_0(p) \qquad \{v^1,\ v^2\} = r_0(p)\, v^1\, v^2 \tag{7}$$

and

$$\{x_i,\ p_j\} = \delta_{ij}\ (1 \leqslant i, j \leqslant N-1) \qquad \{u_0,\ v_0\} = 0 \qquad \{u_0,\ p_i\} = 0 = \{v_0,\ p_i\} \tag{8}$$

where we have introduced $u = u_0\, Q,\ v = Q\, v_0$,

$$r_0(p) = \sum_\alpha \frac{\mathrm{i}}{p_\alpha} (e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha) \tag{9}$$

and α runs over positive roots of $\mathcal{G}$. Namely we have

$$e_\alpha = e_{jk} \quad j < k \quad (e_{ij} e_{kl} = \delta_{kj} e_{il}) \qquad p_\alpha = (p_j - p_k)\,.$$

The variable p (in $r_0(p)$) means that r_0 depends on all moments p_i. The quantum version of the formulae (7)–(9) has been discussed in [6, 7] for the case of $SL_q(2)$ group and, as it has been pointed out in [7], can be postulated for the general case of $SL_q(N)$ in the same form:

$$\begin{aligned} &R_{12}\, u_1\, u_2 = u_2\, u_1\, R(p)_{12} \\ &R(p)_{12}\, v_2\, v_1 = v_1\, v_2\, R_{12} \end{aligned} \tag{10}$$

$$[u_0^1,\ v_0^2] = 0 \qquad [u_0,\ p_i] = 0 = [v_0,\ p_i] \qquad [x_i,\ p_j] = \mathrm{i}\, h\, \delta_{ij} \quad (i, j \leqslant N-1)\,. \tag{11}$$

Here q is a deformation parameter, h is Planck's constant, R_{12} is the well known R-matrix for the $GL_q(N)$ group (see [14, 15]) and we introduce a new R-matrix $R(p)_{12}$ which depends non-trivially on the moments p_i, $\forall\ i$. For simplicity we remove from equations (10) the non-essential factor $(q^{-1/N})$ which transforms the $GL_q(N)$ R-matrix to the $SL_q(N)$ one. Here and below we use R-matrix formalism which was developed in [15]. We note that the u and v algebras (10) can be identified via relation $u = v^{-1}$. Let us recall that the $GL_q(N)$ R-matrix satisfies the YBE

$$\hat{R}\, \hat{R}'\, \hat{R} = \hat{R}'\, \hat{R}\, \hat{R}' \tag{12}$$

and the Hecke relation

$$\hat{R}_{12}^2 = \lambda \hat{R}_{12} + 1 \qquad \lambda = q - q^{-1}\,. \tag{13}$$

where $\hat{R} = \hat{R}_{12} = P_{12} R_{12}$, $\hat{R}' = P_{23} R_{23}$ and P_{12} is a permutation matrix. Using relation (13) we immediately derive from equations (10) that $\hat{R}(p) = \hat{R}(p)_{12} = P_{12}\, R(p)_{12}$ also obeys the Hecke relation

$$\hat{R}(p)^2 = \lambda \hat{R}(p) + 1\,. \tag{14}$$

Considering third-order monomials in u (or in v) and using the commutation relations (10), (11) gives the analogue of the YBE [1, 5–7] for the new objects $R(p_i)_{12}$:

$$(Q_1)^{-1}\, R(p)_{23}\, Q_1\, R(p)_{13}\, (Q_3)^{-1}\, R(p)_{12}\, Q_3 = R(p)_{12}\, (Q_2)^{-1}\, R(p)_{13}\, Q_2\, R(p)_{23}\,. \tag{15}$$

This equation can be rewritten in the form (cf equation (12))

$$\hat{R}(p)\, \widetilde{R}(p)'\, \hat{R}(p) = \widetilde{R}(p)'\, \hat{R}(p)\, \widetilde{R}(p)' \tag{16}$$

where the matrix $\widetilde{R}(p)' = Q_3\, \hat{R}(p)_{23}\, (Q_3)^{-1}$ obviously also satisfies the Hecke condition. In seeking solutions of equations (16) it is convenient to rewrite them as

$$(Q_3^{-1}\, \hat{R}(p)\, Q_3)\, \hat{R}(p)'\, (Q_3^{-1}\, \hat{R}(p)\, Q_3) = \hat{R}(p)'\, (Q_3^{-1}\, \hat{R}(p)\, Q_3)\, \hat{R}(p)' \tag{17}$$

where $\hat{R}(p)' = \hat{R}(p)_{23}$. We call equations (15)–(17) modified or twisted Yang–Baxter equations. Note that from equation (16) one can obtain relations that are similar to the reflection equations:

$$L(p)\,\widetilde{R}(p)'\,L(p)\,\widetilde{R}(p)' = \widetilde{R}(p)'\,L(p)\,\widetilde{R}(p)'\,L(p)$$

$$\widetilde{L}(p)\,\hat{R}(p)\,\widetilde{L}(p)\,\hat{R}(p) = \hat{R}(p)\,\widetilde{L}(p)\,\hat{R}(p)\,\widetilde{L}(p)$$

where $L(p) = \hat{R}(p)^2$, $\widetilde{L}(p) = (\widetilde{R}'(p))^2$.

In the papers [5–7] the explicit form of the matrix $R(p)$ in $SL_q(2)$ case has been presented. There it was also stressed that the elements of the matrix $R(p)$ give the special set of the 6j symbols for the $SL_q(2)$ group. Below we resolve equations (16) (and, thus, derive the explicit formulae for $R(p)$) in the case of the groups $SL_q(N)$ and supergroups $SL_q(N|M)$. Note that special solutions of (17) for arbitrary simple quantum groups (including $SL_q(N)$) are presented in [1]. Our solutions are multiparametric and more general. The specific choices of these parameters lead our solutions to the special forms of $R(p)$ which could be interpreted as the corresponding 6j symbols [5] or as the exchange R-matrix for the vertex algebra [1].

3. Solutions of the modified YBE for the case of linear quantum groups

The mYBE and their solutions for the case of $SL_q(N)$ were first considered in [1]. Here we obtain a more general multiparametric solution for the case of linear quantum groups and supergroups. We will consider the case of $GL_q(N)$ and $GL_q(K|N-K)$. In the case of special q-groups, the solutions $R(p)$ can be obtained by multiplying $R(p)$ by some factor which is a simple function of q (see below).

Let us seek a solution of the mYBE (17) in the form

$$\hat{R}_{12} = \hat{R}(p)^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\,a_{i_1 i_2}(p) + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\,b_{i_1 i_2}(p)\,. \tag{18}$$

Without limiting generality one can put $b_{ii}(p) = 0$. Now the condition that the $R(p)$ (taking the form (18)), satisfying the Hecke relation (14) gives the following constraints:

$$b_{ij} + b_{ji} = \lambda \qquad i \neq j \tag{19}$$

$$a_{ij}\,a_{ji} - b_{ij}\,b_{ji} = 1 \qquad i \neq j \tag{20}$$

$$a_i^2 - \lambda a_i - 1 = 0 \Rightarrow a_i - \lambda = \frac{1}{a_i} \qquad a_i \equiv a_{ii}\,. \tag{21}$$

Note that equation (21) has two solutions: $a_i = \pm q^{\pm 1}$ and therefore the coefficients a_i are independent of the parameters p_k. If we take $a_i = q,\ \forall\, i$ (or $a_i = -q^{-1},\ \forall\, i$) then we will have the case of the group $GL_q(N)$ (or $GL_{-q^{-1}}(N)$). But if we consider the mixing case, $a_i = q$ for $1 \leqslant i \leqslant K$ and $a_i = -q^{-1}$ for $K+1 \leqslant i \leqslant N$, then we come to the case of the supergroups $GL_q(K|N-K)$.

Now let us use the relations (cf equation (11))

$$\exp(-\mathrm{i}\,x_j)\,p_k\,\exp(\mathrm{i}\,x_j) = p_k + h\delta_{kj} \qquad (1 \leqslant k, j \leqslant N) \tag{22}$$

and substitute (18) in the mYBE (17). As a result, in addition to relations (19)–(21), we obtain new constraints on the functions $a_{ij}(p)$ and $b_{ij}(p)$. First of all we deduce

$$a_{ij}(p_1, \ldots, p_N) = a_{ij}(p_i, p_j) \qquad b_{ij}(p_1, \ldots, p_N) = b_{ij}(p_i, p_j) \tag{23}$$

and, as a consequence, relations (19), (20) have to be rewritten in the form

$$b_{ij}(p_i,p_j)+b_{ji}(p_j,p_i)=\lambda$$

$$a_{ij}(p_i,p_j)\,a_{ji}(p_j,p_i)-b_{ij}(p_i,p_j)\,b_{ji}(p_j,p_i)=1\,.$$

Then we have the constraint

$$b_{ij}\,b_{jk}\,b_{ki}+b_{ik}\,b_{kj}\,b_{ji}=0\qquad i\neq j\neq k\neq i \tag{24}$$

(where there is no summation over i, j, k) and the equations

$$b_{ij}(p_i+h,p_j)=\frac{b_{ij}(p_i,p_j)\,a_i}{1/a_i+b_{ij}(p_i,p_j)} \tag{25}$$

$$b_{ij}(p_i,p_j+h)=\frac{b_{ij}(p_i,p_j)/a_j}{a_j-b_{ij}(p_i,p_j)}\,. \tag{26}$$

Using equations (25) and (26) leads to the following general relations:

$$\begin{aligned} b_{ij}(p_i+n\,h,p_j+m\,h) &= \frac{a_i^n\,a_j^{-m}\,b_{ij}(p_i,p_j)}{a_i^{-n}\,a_j^m+b_{ij}(p_i,p_j)\,(a_i^n a_j^{-m}-a_i^{-n}a_j^m)/\lambda} \\ &= \frac{\lambda\,a_i^n\,a_j^{-m}\,b_{ij}(p_i,p_j)}{a_i^n\,a_j^{-m}\,b_{ij}(p_i,p_j)+a_i^{-n}\,a_j^m\,b_{ji}(p_j,p_i)}\,. \end{aligned} \tag{27}$$

From these relations one can immediately find the general solution for the coefficients $b_{ij}(p)$:

$$\begin{aligned} b_{ij}(p_i,p_j) &= \frac{a_i^{p_i/h}\,a_j^{-p_j/h}\,b_{ij}^0}{a_i^{-p_i/h}\,a_j^{p_j/h}+b_{ij}^0\,(a_i^{p_i/h}a_j^{-p_j/h}-a_i^{-p_i/h}a_j^{p_j/h})/\lambda} \\ &= \frac{\lambda\,a_i^{p_i/h}\,a_j^{-p_j/h}\,b_{ij}^0}{a_i^{p_i/h}\,a_j^{-p_j/h}\,b_{ij}^0+a_i^{-p_i/h}\,a_j^{p_j/h}\,b_{ji}^0} \end{aligned} \tag{28}$$

where the constants $b_{ij}^0=b_{ij}(0,0)$ have to obey the algebraic relations

$$\begin{aligned} &b_{ii}^0=0\qquad b_{ij}^0+b_{ji}^0=\lambda \\ &b_{ij}^0\,b_{jk}^0\,b_{ki}^0+b_{ik}^0\,b_{kj}^0\,b_{ji}^0=0 \end{aligned} \tag{29}$$

which can be deduced by the substitution of equation (28) in (19) and (24).

It is now clear that if $a_i=a_j$ (the indices i and j 'have the same grading') then $b_{ij}(p_i,p_j)=b_{ij}(p_i-p_j)$, but if $a_i=-1/a_j$ (the case of supergroups when the indices i and j 'have opposite grading') then we deduce that $b_{ij}(p_i,p_j)=b_{ij}(p_i+p_j)$. Note that the only conditions on the parameters $a_{ij}(p)$ needed for the solution of the mYBE are listed in (20) and (23).

Let us consider the case of the $GL_q(N)$ group. In this case we have $a_i=q\ \forall\ i$ and relation (28) takes the form

$$b_{ij}(p_i-p_j)=\frac{q^{(p_i-p_j)/h}\,b_{ij}^0}{q^{(-p_i+p_j)/h}+[\frac{p_i-p_j}{h}]_q\,b_{ij}^0} \tag{30}$$

while for the functions $a_{ij}(p_i,p_j)$ we obtain the following relations from (20):

$$a_{ij}(p_i,p_j)\,a_{ji}(p_j,p_i)=1+\frac{\lambda^2\,b_{ij}^0\,b_{ji}^0}{(q^{(p_i-p_j)/h}\,b_{ij}^0+q^{(p_j-p_i)/h}\,b_{ji}^0)^2}\,. \tag{31}$$

In equation (30) we have used the standard notation $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. One can obtain from these expressions the solutions discussed in [1, 5–7] if we take the proper normalization of a_{ij} (e.g., Faddeev's or unitary normalization $a_{ij}(p_i - p_j) = a_{ji}(p_j - p_i)$) and consider, in (30), (31), the limit $b^0_{ij} \to \infty$ $(i < j)$. This limit can be performed self-consistently such that it does not cancel the conditions (29). Then we recall that our consideration was done for the case of the general groups $GL_q(K|N-K)$. In fact one can obtain SL-matrices $R(p)$ multiplying them by the functions $q^{1/(K-N)-1/K}$ which are needed for obtaining the identity $Sdet_q(R) = 1$ (for the $SL_q(N)$ case we have to multiply $R(p)$ by $q^{-1/N}$). As a result for the $SL_q(N)$ case we have the matrix $[q^{-1/N} \cdot R(p)]$, equation (18), with the substitution

$$b_{ij}(p_i - p_j) = \frac{q^{(p_i - p_j)/h}}{[(p_i - p_j)/h]_q} = \lambda - b_{ji}(p_j - p_i) \tag{32}$$

$$a_{ij}(p_i - p_j) = \frac{\left([(p_i - p_j)/h + 1]_q\,[(p_i - p_j)/h - 1]_q\right)^{1/2}}{[(p_i - p_j)/h]_q\,\epsilon_{ij}} = a_{ji}(p_j - p_i) \tag{33}$$

where $\epsilon_{ij} = \pm 1$ $(i < j)$, $= \mp 1$ $(j > i)$. This choice of a_{ij} leads to the unitary condition (for real q and $p_i^\dagger = p_i$):

$$R(p)^\dagger_{12} = R(p)^{t_1 t_2}_{12} = R(p)_{21}.$$

Analogously, for the $SL_q(K|N-K)$ case, we obtain the matrix $q^{1/(K-N)-1/K} \cdot R(p)$, equation (18):

$$\begin{aligned}\hat{R}(p)_{SL_q(K|N-K)} = q^{1/(K-N)-1/K}\Big(&\delta^{i_1}_{j_2}\,\delta^{i_2}_{j_1}\big[(-1)^{(i_1)}\,q^{1-2(i_1)}\delta^{i_1 i_2} + a(p_i - p_j)(\theta_{K+1,i}\,\theta_{K+1,j}\\ &+\theta_{i,K}\theta_{j,K}) + a(p_i + p_j)(\theta_{K+1,i}\,\theta_{j,K} + \theta_{i,K}\theta_{K+1,j})\big]\\ &+\delta^{i_1}_{j_1}\,\delta^{i_2}_{j_2}\big[b(p_i - p_j)\theta_{K+1,i}\,\theta_{K+1,j} + b(p_j - p_i)\theta_{i,K}\,\theta_{j,K}\\ &b(p_i + p_j)\theta_{K+1,i}\,\theta_{j,K} + b(-p_i - p_j)\theta_{i,K}\,\theta_{K+1,j}\big]\Big)\end{aligned} \tag{34}$$

where $(i) = 0$ for $1 \leqslant i \leqslant K$ and $= 1$ for $K + 1 \leqslant i \leqslant N$, $\theta_{ij} = 1$ for $i > j$ and $= 0$ for $i \leqslant j$. The functions $a(p_i - p_j) = a_{ij}(p_i - p_j)$, $b(p_i - p_j) = b_{ij}(p_i - p_j)$ are defined in (32), (33).

To conclude this section we stress that we have found a more general solution of the mYBE, equations (15)–(17), than that obtained in [1, 5]. Namely, our solutions depend on the set of arbitrary parameters b^0_{ij} constrained by the conditions (29). The role of these parameters is still to be clarified. Taking some special limits and choosing the normalization of a_{ij} one leads to the known solutions $R(p)$ of the papers [1] and [5] (see, e.g., equations (32), (33)).

4. Modified (twisted) YBE with spectral parameters

In this section we show that mYBE, equations (15)–(17), can be generalized by introducing spectral parameters $y, z, \ldots$. We demonstrate that every solution $R(p)$ which has been found in section 3 will lead to the solution $R(p, y)$ for the mYBE with spectral parameters. It is interesting to note that such introduction of the spectral parameters can be done in complete analogy with the usual method of obtaining the trigonometric solutions

$$\hat{R}(y) = y^{-1}\,\hat{R} - y\,\hat{R}^{-1} \tag{35}$$

of the YBE from the R-matrices $\hat{R}$ related to the $GL_q(N)$ groups. On the other hand, we know that the trigonometric solutions (35) are related to the quantum Kac–Moody algebras [16]. In this connection (following the statements of [5–7]) it is natural to conjecture that $R(p, y)$ could be interpreted as a special set of 6j symbols for the q-deformations of linear affine algebras.

The natural assumption about the form of the mYBE dependent on the spectral parameters is as follows

$$\hat{R}(p, y)\, \widetilde{R}'(p, y \cdot z)\, \hat{R}(p, z) = \widetilde{R}'(p, z)\, \hat{R}(p, y \cdot z)\, \widetilde{R}'(p, y)\,. \tag{36}$$

Now it is not difficult to check by using mYBE (16) and the Hecke relations (14) that the following matrices:

$$\begin{aligned} \hat{R}(p, y) &= y^{-1}\, \hat{R}(p) - y\, \hat{R}(p)^{-1} \\ \widetilde{R}'(p, y) &= y^{-1}\, \widetilde{R}'(p) - y\, (\widetilde{R}'(p))^{-1} = Q_3\, \hat{R}'(p, y)\, Q_3^{-1} \end{aligned} \tag{37}$$

are the solutions of the new mYBE (36). We note that the solutions (37) satisfy the identity

$$\hat{R}(p, y)\, \hat{R}(p, y^{-1}) = \left(\lambda^2 - (y - y^{-1})^2\right)$$

which is a kind of unitary condition for $R(p)$ (if $y^* = y^{-1}$). It is clear that the analogue of relations (10) depending on the spectral parameters has the form

$$\hat{R}(y)\, u_1(yz)\, u_2(z) = u_1(z)\, u_2(yz)\, \hat{R}(p, y)\,.$$

Now let us put $q = \exp(\gamma\, h)$ [6, 7] and $y = \exp(-\frac{1}{2}\lambda\, \theta)$. Following [6] we consider two different cases: the deformed classical case ($h = 0,\ \gamma \neq 0$) and the undeformed quantum case ($\gamma = 0,\ h \neq 0$). In the first case we obtain the result that $\hat{R}(p, y)/\lambda$ tends to the Yangian R-matrix $\hat{R}(\theta) = \theta\, P_{12} - 1$ which satisfies the usual YBE

$$\hat{R}(\theta)\, \hat{R}'(\theta + \theta')\, \hat{R}(\theta') = \hat{R}'(\theta')\, \hat{R}(\theta + \theta')\, \hat{R}'(\theta)\,.$$

In the second case we derive

$$\lim_{\gamma \to 0} \frac{\hat{R}(p, y)}{\lambda} = \hat{R}(p, \theta) = \theta\, \hat{R}^0(p) - 1 \tag{38}$$

where $\hat{R}^0(p)$ is represented in the form (18) with the following parameters (cf [6, 7]):

$$b_{ij} = \frac{h}{p_i - p_j} = -b_{ji}$$

$$a_{ij} = \frac{((p_i - p_j + h)(p_i - p_j - h))^{1/2}}{\epsilon_{ij}(p_i - p_j)} = a_{ji}$$

and the matrix $\hat{R}(p, \theta)$, equation (38), satisfies the twisted YBE

$$\hat{R}(p, \theta)\, \widetilde{R}'(p, \theta + \theta')\, \hat{R}(p, \theta') = \widetilde{R}'(p, \theta')\, \hat{R}(p, \theta + \theta')\, \widetilde{R}'(p, \theta)\,. \tag{39}$$

To conclude this paper we note that it would be extremely interesting to use the twisted YBEs (36), (39) and their solutions (37), (38) to formulate integrable models, e.g., via the box construction of [2] or to relate these solutions to the braiding matrices describing generalized statistics (see, e.g., [17]).

Acknowledgments

The author would like to thank G Arutyunov, R Kashaev for helpful comments and M Mintchev for valuable discussions and the idea of introducing spectral parameter in the twisted YBE. I am grateful to A Di Giacomo and M Mintchev for hospitality at the Physics Department of the University of Pisa where the work published in this paper was carried out. This work was partially supported by the exchange program between INFN and JINR (Dubna), INTAS (grant No 93-127) and RFFI (grant 95-02-05679-a).

Note added. After submitting this paper for publication the author was informed about the papers [18] where equations (39) were proposed and their elliptic solutions found, and [19] where these solutions have been applied to quantize Calogero–Moser-type models.

References

[1] Cremer E and Gervais J-L 1990 *Commun. Math. Phys.* **134** 619
Gervais J-L and Neveu A 1984 *Nucl. Phys.* B **238** 125
Bilal A and Gervais J-L 1989 *Nucl. Phys.* B **318** 579
[2] Kashaev R M and Stroganov Yu G 1993 *Mod. Phys. Lett.* **8A** 2299
[3] Bazhanov V and Baxter R 1992 *J. Stat. Phys.* **69** 453
Kashaev R, Mangazeev V and Stroganov Yu G 1993 *Int. J. Mod. Phys.* A **8** 1399
[4] Mangazeev V and Stroganov Yu G 1993 *Mod. Phys. Lett.* **8A** 3475
[5] Faddeev L D 1990 *Commun. Math. Phys.* **132** 131
[6] Alekseev A Yu and Faddeev L D 1991 *Commun. Math. Phys.* **141** 413
[7] Bytsko A G and Faddeev L D 1995 The q-analogue of model space and CGC generating matrices *Preprint* q-alg/9508022
[8] Sklyanin E K 1994 *Algebra i Analiz* **6** (6) 227
[9] Avan J, Babelon O and Talon M 1994 *Algebra i Analiz* **6** (6) 67
[10] Alvarez-Gaume L, Gomez C and Sierra G 1989 *Phys. Lett.* **220B** 142; 1990 *Nucl. Phys.* B **330** 347
Moore G and Reshetikhin N Yu 1990 *Nucl. Phys.* B **238** 557
[11] Drinfeld V G 1989 *Algebra i Analiz* **1** 114
[12] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313
Perelomov A M 1990 *Integrable Systems in Classical Mechanics and Lie Algebras* (Moscow: Nauka)
[13] Alekseev A Yu and Todorov I T 1994 *Nucl. Phys.* B **421** 413
[14] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[15] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1989 *Algebra i Analiz* **1** (1) 178 (Engl. transl. 1990 *Leningrad Math. J.* **1** 193
[16] Drinfeld V G 1986 Quantum groups *Proc. ICM* vol 1 (Berkeley, CA: Academic) p 798
[17] Liguori A and Mintchev M 1995 *Commun. Math. Phys.* **169** 635
Liguori A, Mintchev M and Rossi M 1995 *Lett. Math. Phys.* **35** 163
[18] Felder G 1994 Conformal field theory and integrable systems associated to elliptic curves *Preprint* hep-th/9407154; 1995 Elliptic quantum groups *Proc. XIth Int. Congr. of Mathematical Physics (Paris, July 1994)* (Boston: International) p 211
[19] Avan J, Babilon O and Billey E 1995 The Gervais–Neveu–Felder equation and the quantum Calogero–Moser systems *Preprint* LPTHE 95-25, hep-th/9505091